# FALL 2024: MATH 790 HOMEWORK

Homework problems from *Linear Algebra Done Right* will be labelled LADR.

**HW 1.** Let $V$ be a vector space over the field $F$. Note that we are not assuming that $V$ is finite dimensional.

(i) Prove the following version of the exchange property. Let $\{u_1, \dots, u_n\} \subseteq V$ and set $U := \langle u_1, \dots, u_n \rangle$. Suppose $v_1, \dots, v_m \in U$ are linearly independent. Prove that $m \leq n$.
(ii) Give a detailed prove using Zorn's lemma to show that any vector space has a basis.
(iii) Assume that $F = \mathbb{C}$. Prove that $V$ is also a vector space over $\mathbb{R}$, and assuming $V$ is finite dimensional over $\mathbb{C}$, find the dimension of $V$ as a vector space over $\mathbb{R}$ in terms of the dimension of $V$ over $\mathbb{C}$.
(iv) Let $W \subseteq V$ be a subspace. Use Zorn's lemma to prove there exists a subspace $U \subseteq V$ maximal with respect to the property that $W \cap U = 0$.

**HW 2.** Let $V$ be a vector space over the field $F$.

(i) Suppose $T : V \to W$ is a linear transformation between finite dimensional vectors spaces. Assume $\alpha_1, \alpha_2$ are bases for $V$ and $\beta_1, \beta_2$ are bases for $W$. Use the crucial formula from the lecture of August 28 to write a formula relating the matrices $[T]_{\alpha_1}^{\beta_1}$ and $[T]_{\alpha_2}^{\beta_2}$.
(ii) Prove that if the dimension of $V$ equals $n$, with $n > 0$, then there cannot exist a chain of subspaces $(0) \subsetneq W_1 \cdots \subsetneq W_n \subsetneq V$. Conclude that if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is an ascending chain of subspaces of $V$, then there exists $n_0 \geq 1$ such that $U_s = U_{n_0}$, for all $s \geq n_0$.
(iii) Suppose $F$ is infinite. Prove that $V$ is not the union of finitely many proper subspaces of $V$.

**HW 3.** This homework uses the notation from the second day of class, as it appears in the Daily Update from August 28. Let $A$ be an $n \times n$ matrix with coefficients in the field $F$.

(i) Let $T : V \to W$ be a linear transformation and set $A = [T]_\alpha^\beta$. For $v \in V$ let $[v]_\alpha$ denote the $n \times 1$ column vector in $F^n$ obtained as follows: If $v = a_1 v_1 + \cdots + a_n v_n$, then $[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. The vector $[T(v)]_\beta$ in $W$ is defined similarly. Prove that $[T(v)]_\beta = A \cdot [v]_\alpha$.
(ii) Show that if $B$ is obtained from $A$ by interchanging two rows, then $|B| = -|A|$.
(iii) Let $E$ be an elementary matrix, i.e., an $n \times n$ matrix obtained from $I_n$ by applying an elementary row operation. Prove that $EA$ is obtained from $A$ by apply the same elementary row operation to $A$.
(iii) Prove that if $A$ is any matrix, then there is a sequence of elementary row operations that put $A$ into *reduced row echelon form.*
(iv) Show that if $E$ is an elementary matrix corresponding to an elementary row operation of a given type, then $E^t$ is an elementary matrix corresponding to a row operation of the same type.

**HW 4.** For problems these you may use any of the properties of the determinant discussed in class.

(i) For an $n \times n$ matrix $A$, verify the Laplace expansion along the $k$th row: $|A| = \Sigma_{j=1}^n (-1)^{j+k} a_{kj} |A_{kj}|$.
(ii) Let $A$ be an $n \times n$ invertible matrix such that every entry is $\pm 1$. Prove that $|A|$ is an integer divisible by $2^{n-1}$.
(iii) Suppose that $A$ and $B$ are $(2k+1) \times (2k+1)$ matrices over $\mathbb{R}$ such that $AB = -BA$. Prove that $A$ and $B$ cannot both be invertible.

**HW 5.** Let $V$ be a vector space of dimension $n$ over the field $F$.

(i) Prove that the vector spaces $\mathcal{L}(V, V)$ and $\mathrm{M}_n(F)$ are isomorphic.
(ii) Using the Cayley-Hamilton theorem for matrices, prove that $\chi_T(T) = 0$, for $T \in \mathcal{L}(V, V)$.
(iii) For $f(x) \in F[x]$, with $s$ the degree of $f(x)$, prove that $|xI_s - C(f(x))| = f(x)$, where $C(f(x))$ is the companion matrix of $f(x)$. In other words $\chi_{C(f(x))}(x) = f(x)$.

**HW 6.** In the first two problems below, $V$ is a vector space of dimension $n$ and $B := \{v_1, \ldots, v_n\} \subseteq V$ is a basis for $V$.

1. For $v \in V$, write $v = a_1v_1 + \cdots + a_nv_n$. Define $[v]_B := \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, a column vector in $F^n$.

   (i) For $\alpha, \beta \in F$ and $v, w \in V$, show that $[\alpha v + \beta w]_B = \alpha[v]_B + \beta[w]_B$.

   (ii) For $T \in \mathcal{L}(V, V)$ and $v \in V$, show that $[T(v)]_B = [T]_B^B \cdot [v]_B$.

2. Suppose $\lambda \in F$ and $T \in \mathcal{L}(V, V)$. Using the corresponding result for matrices, prove that $\lambda$ is an eigenvalue of $T$ if and only if $\chi_T(\lambda) = 0$.

3. Prove a uniqueness statement for the division algorithm in $F[x]$, i.e., prove that if $f(x), g(x), h(x), r(x), h_0(x), r_0(x)$ are in $F[x]$ and

$$g(x) = f(x)h(x) + r(x) = f(x)h_0(x) + r_0(x),$$

where $r(x)$, $r_0(x)$ are either zero or have degree less than the degree of $f(x)$, then $h(x) = h_0(x)$ and $r(x) = r_0(x)$.

**HW 7.** 1. Let $W \subseteq \mathbb{R}$ be a plane through the origin and $L \subseteq \mathbb{R}^3$ a line through the origin, with $L \not\subseteq W$. Prove that $\mathbb{R}^3 = W \oplus L$.

2. Suppose $V = W_1 + W_2$, for proper non-zero subspaces $W_1, W_2 \subseteq V$. Prove there exists a subspace $U_2 \subseteq W_2$ such that $V = W_1 \oplus U_2$.

3. Suppose $D$ is an $n \times n$ diagonal matrix with diagonal entires $\lambda_1, \ldots, \lambda_n$. Prove that $\lambda_1, \ldots, \lambda_n$ are the only eigenvalues of $D$.

**HW 8.** 1. Suppose $B' = \{v_1', \ldots, v_n'\}$ is a basis for $V$ and $P = (p_{ij})$ an $n \times n$ matrix over $F$. Consider $B = \{v_1, \ldots, v_n\}$, where each $v_i = p_{i1}v_1' + \cdots + p_{in}v_n'$. Show that $B$ is a basis for $V$ if and only if $P$ is an invertible matrix.

2. Transcribe the main theorem concerning diagonalizability of linear transformations presented at the end of the lecture on September 13 to a statement about diagonalizability for matrices, and then use the linear transformation form of the theorem to prove the matrix form of the theorem.

**HW 9.** 1. Suppose $B$ is a basis for the vector space $V$ and $B = B_1 \cup \cdots \cup B_r$ is a partition of $B$. Set $W_i := \text{Span}(B_i)$, for each $1 \leq i \leq r$. Show that $V = W_1 \oplus \cdots \oplus W_r$. Note: $V$ need not be finite dimensional, though you can assume this initially, to get a feeling for how this works.

2. Let $T : F^n \to F^n$ be a linear transformation, suppose $E \subseteq F^n$ is the standard basis, and write $A = [T]_E^E$. Suppose $P$ is an invertible matrix such that $P^{-1}AP = D$, where $D$ is a diagonal matrix. Let $C_1, \ldots, C_n$ be the columns of $P$, and set $B := \{C_1, \ldots, C_n\}$. Prove that $B$ is a basis for $F^n$ and $[T]_B^B = D$.

3. Let $F$ be a field and $T_A : F^2 \to F^2$ be the linear transformation whose matrix with respect to the standard basis is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Determine if $T_A$ is diagonalizable over the fields: (a) $F = \mathbb{R}$, (b) $F = \mathbb{C}$, (c) $F = \mathbb{Z}_2$, and (d) $F = \mathbb{Z}_3$.

4. Let $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis is $B = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}$. Show that $T_B$ is diagonalizable. Find an invertible $2 \times 2$ matrix $P$ such that $P^{-1}BP$ has the eigenvalues of $B$ down its diagonal.

Henceforth, whenever we say that $V$ is an inner product space, we assume that $F = \mathbf{C}$ or $\mathbb{R}$.

**HW 10.** 1. Let $V$ denote the vector space of complex polynomials having degree less than or equal to $n$. For $f, g \in V$, set $\langle f(x), g(x) \rangle := \int_{-1}^{1} f(x)\overline{g(x)}\, dx$. Show that this defines an inner product on $V$.

2. Suppose $V$ is an inner product space Show:

   (i) $||v|| = 0$ if and only if $v = \vec{0}$.

   (ii) $||\lambda v|| = |\lambda| \cdot ||v||$, for all $v \in V$ and $\lambda \in F$. Here for the complex number $\lambda$, $|\lambda| := \sqrt{a^2 + b^2}$, if $\lambda = a + bi$. Note if $\lambda \in \mathbb{R}$, $|\lambda|$ is just the absolute value of $\lambda$.

(iii) If $\vec{0} \neq v \in V$, and $\lambda = \frac{1}{||v||}$, show that $||\lambda v|| = 1$.

3. Suppose $V$ is an inner product space defined over $\mathbb{R}$. Show:
   (i) $\langle u+v, u-v\rangle = ||u||^2 - ||v||^2$, for all $u, v \in V$.
   (ii) If $u, v \in V$ have the same length, then $u+v$ is orthogonal to $u-v$.
   (iii) The two diagonals of any rhombus are perpendicular to each other.

**HW 11.** 1. Let $V$ be the vector space of real polynomials of degree less than or equal to two. Define $\langle f(x), g(x)\rangle := \int_0^2 f(x)g(x)$. Find an orthonormal basis for $V$.

2. Let $W$ be a subspace of the finite dimensional inner product space $V$. Show that $V = W \oplus W^{\perp}$. Hence the name *orthogonal complement* for $W^{\perp}$. Hint: Start with an orthogonal basis for $W$ and extend it to an orthogonal basis for $V$. Thus, every vector $v \in V$ can we written uniquely as $v = w + w'$, with $w \in W$ and $w' \in W^{\perp}$. The vector $w$ is called the *orthogonal projection* of $v$ onto $W$.

**HW 12.** 1. Let $C = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$. Find an orthogonal matrix $Q$ such that $Q^{-1}CQ$ is diagonal. Now, Let $T_C : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{R}^3$ is $C$. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for $T$.

2. Find a $2 \times 2$ matrix over $\mathbb{R}$ that is diagonalizable, but not orthogonally diagonalizable.

**HW 13.** Let $V$ be a finite dimensional inner product space over $\mathbf{C}$.

1. For $T \in \mathcal{L}(V, V)$, using the definition of $T^*$ from class, prove a version of the first spectral theorem over $\mathbf{C}$ for $T$, using the corresponding result for matrices.

2. For $T, S \in \mathcal{L}(V, V)$, use the definition of $T^*$ from class to prove the following properties:
   (i) $(S+T)^* = S^* + T^*$.
   (ii) $(ST)^* = T^*S^*$.
   (iii) $\lambda T)^* = \overline{\lambda} T^*$.
   (iv) $(T^*)^* = T$.
   (v) If $T$ is invertible, $(T^{-1})^* = (T^*)^{-1}$.

**HW 14.** Give an example of a matrix $A \in \mathrm{M}_2(\mathbf{C})$ that is not self-adjoint, but $A$ is normal, and its entries are not in $\mathbb{R}$. Then show that, for you particular choice of $A$, $||Av|| = ||A^*v||$, for all $v \in \mathbf{C}^2$.

**HW 15.** Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Show that these are normal matrices. Then show that when $A$ acts on $\mathbb{R}^4$ by multiplication, there are four invariant subspaces, whereas when $B$ acts on $\mathbb{R}^4$ by multiplication, there are infinitely many invariant subspaces. The difference here lies in the difference between $\mu_A(x)$ and $\mu_B(x)$. Calculate these polynomials.

**HW 16.** 1. Let $A$ be an $m \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$. Prove that: (a) $A^*A$ and $AA^*$ have the same eigenvalues, counted with multiplicity and (b) $A^*A$ and $A$ have the same rank.

2. Find the singular value decomposition for the following matrices: (a) $A = \begin{pmatrix} i & 2i \\ 3i & 6i \end{pmatrix}$; (b) $B = \begin{pmatrix} 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

**HW 17.** Let $F[x]$ denote the ring of polynomials with coefficients in the field $F$.

(i) Let $p(x) \in F[x]$ be a non-constant irreducible polynomial. Prove that for any non-constant $f(x)$ in $F[x]$, the GCD of $p(x), f(x)$ is either $p(x)$ or 1.

(ii) Show that if $p(x)$ is irreducible over $F$ and $p(x)$ divides $f(x) \cdot g(x)$, then $p(x)$ divides $f(x)$ or $p(x)$ divides $g(x)$. (Hint: Use (i) and Bezout's Principle.)

(iii) Prove that if $p_1(x) \cdots p_r(x) = q_1(x) \cdots q_s(x)$, and each $p_i(x), q_j(x)$ is monic and irreducible over $F$, then $r = s$, and after re-indexing, $q_i(x) = \cdot p_i(x)$. In other words, the factorization property for polynomials in $F[x]$ is in fact a *unique factorization* property.

**HW 18.** 1. Consider $f(x) = x^4 + x^3 + x + 1$ and $x^4 + 2x$ in $\mathbb{Z}_2[x]$. Use the Euclidean algorithm to find the GCD of $f(x)$ and $g(x)$, then write this GCD as $a(x)f(x) + b(x)g(x)$, for some $a(x), b(x) \in \mathbb{Z}_2[x]$.

2. Consider the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ as an element of $\mathrm{M}_2(\mathbb{R})$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$. Find a basis $B \subseteq \mathbb{R}^3$ such that the matrix of $T$ with respect to $B$ is block diagonal, with one block a $2 \times 2$ companion matrix and the other block a $1 \times 1$ matrix.

3. Suppose $A$ is a $3 \times 3$ matrix over $\mathbb{R}$ whose minimal polynomial equals $x^3$. Show there is an invertible matrix $P$ such that $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hint: What form must $\mu_{A,v}(x)$ take, for $v \in \mathbb{R}^3$.

**HW 19.** 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis is $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}$.

(i) For $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, show that $\langle T, v \rangle = \mathbb{R}^3$.

(ii) For $w = (T^2 + I)(v)$, find $\mu_{T,w}(x)$ and determine $\langle T, w \rangle$,

2. Suppose $T \in \mathcal{L}(V, V)$ and there are non non-trivial $T$-invariant subspaces of $V$. Prove $V$ is a $T$-cyclic vector space.

3. Assume $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $\langle T, v \rangle = \mathbb{R}^3$, for some $v \in \mathbb{R}^3$. Let $N$ be the number of $T$-cyclic subspaces. Show that $N = 2, 4$, or 6.

**HW 20.** 1. Let $A \in \mathrm{M}_3(\mathbb{R})$. Show that $\mu_A(x)$ cannot be an irreducible polynomial of degree two.

2. Let $E = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$ be the standard basis and suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $[T]_E^E = \begin{pmatrix} -1 & 3 & -2 \\ -1 & 3 & -4 \\ -1 & 1 & -2 \end{pmatrix}$.

(i) Find $\mu_{T,e_i}(x)$, for each $e_1, e_2, e_3$.

(ii) Compute $\mu_T(x)$.

(iii) Find a maximal vector for $\mathbb{R}^3$ with respect to $T$.

3. Do the same as in 3, for the matrix $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

**HW 21.** 1. Use the invariant factor form of the RCF theorem to prove the elementary divisor form of the RCF theorem, as stated in the lecture of October 21.

2. Prove a matrix version of the elementary divisor form of the RCF theorem.

**HW 22.** For the two matrices given in HW 20, find their rational canonical forms and the corresponding change of basis matrices.

**HW 23.** 1. For $p \geq 1$, find $p$ distinct $p$th roots of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2. Find the solution to the system of first order linear differential equations given by the vector equation $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, with initial condition $\mathbf{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Here $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

**HW 24.** Prove the third isomorphism theorem: Let $U, W$ be subspaces of the vector space $V$. Prove that $(U + W)/W$ is isomorphic to $U/(U \cap W)$. Hint: Find a well-defined surjective linear transformation from $U \to (U + W)/W$ and then apply the First Isomorphism Theorem.

2. Let $V$ and $U$ be vector spaces and $W \subseteq V$ a subspace. Set $K := \{f \in \mathcal{L}(V, U) \mid W \subseteq \text{kerne}(f)\}$. Show that $K$ is a subspace of $\mathcal{L}(V, U)$ and $\mathcal{L}(V, U)/K \cong \mathcal{L}(V/W, U)$.

**HW 25.** 1. Given vector spaces $V, W$, suppose $(P, f)$ is a tensor product of $V$ and $W$. Suppose $\alpha : P \to P_1$ is an isomorphism of vector spaces. Set $f_1 := \alpha \circ f$. Show that $(P_1, f_1)$ is a tensor product of $V$ and $W$.

2. Let $L, M$ be vector spaces over $F$. Suppose that $T : L \to M$ and $S : M \to L$ are linear transformations sch that $ST$ is the identity on $L$ and $TS$ is the identity on $M$. Prove that $T$ is an isomorphism with inverse $S$.

3. For vector spaces $V, W_1, W_2$ over $F$ prove that $V \otimes (W_1 \oplus W_2) \cong (V \otimes W_1) \oplus (V \otimes W_2)$.

**HW 26.** Prove the properties (a), (b), (c) for direct sums stated at the end of the lecture on Friday, November 22.

**HW 27.** Let $V$ be a vector space over $F$. Let $L$ denote $\text{Span}\{v \otimes v' - v' \otimes v \mid v, v' \in V\} \subseteq V \otimes V$. Let $v_1 * v_2$ denote the coset $v_1 \otimes v_2 + L$ in the quotient space $(V \otimes V)/L$. Set $S^2(V) := (V \otimes V)/L$, the *symmetric square* of $V$.

(i) Show that the same bilinear properties holding in $V \otimes V$ hold with respect to the property $*$ in $S^2(V)$.

(ii) Show that $v_1 * v_2 = v_2 * v_1$ in $S^2(V)$, for all $v_1, v_2 \in V$.

(iii) Suppose $v_1, \ldots, v_n$ is a basis for $V$. Find a basis for $S^2(V)$.

(iv) If $\dim(V) = n$, what is $\dim(S^2(V))$ ?

(v) Given a vector space $U$, a bilinear map $h : V \times V \to U$ is *symmetric* if $h(v_1, v_2) = h(v_2, v_1)$ for all $v_1, v_2 \in V$. Let $\hat{f} : V \times V \to S^2(V)$ be the natural map i.e., the usual bilinear map $f : V \times V \to V \otimes V$ followed by the quotient map from $V \otimes V \to S^2(V)$. Prove that $\hat{f}$ is a symmetric bilinear map, and given any vector space $U$ and a symmetric bilinear map $g : V \times V \to U$, there exists a unique linear transformation $T : S^2(V) \to U$ such that $T \circ \hat{f} = g$.

**HW 28.** Let $G$ be the matrix group of $3 \times 3$ permutation matrices. A $G$-invariant subspace $U \subseteq \mathbf{C}^3$ is *irreducible* is it has no proper $G$-invariant subspaces. Show that $\mathbf{C}^3$ is the direct sum of two irreducible $G$-invariant subspaces.